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THE STRUCTURE OF A VISCOUS FLUID FLOW NEAR A ROTATING DISK EDGE

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V. P. SHIDLOVSKII
(Moscow)

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The problem formulated by Kármán for the laminar flow of incompressible fluid subjected to the action of a uniformly rotating disk [1] has a reliable numerical solution on condition that the disk radius is infinitely large [2]. Its extension to the case of a disk of finite radius and fairly considerable Reynolds numbers is best carried out by the method of asymptotic analysis which is used in investigations of flow near the trailing edge of a flat plate [3, 4]. Structure of the trail close to the disk edge and the order of dimensions of the transition region, which must be introduced in order to satisfy all boundary conditions, are established in this manner. The problem of flow in that region reduces to solving equations of the three-dimensional boundary layer with special boundary conditions. The asymptotic behavior of related equations is studied as the regions of basic flow and of the close trail is gradually approached, its principal correction term in the expression for the coefficient of the moment of friction forces is determined for a disk of finite dimensions.

1. Let us consider the steady flow of a viscous fluid induced by a disk of radius R, rotating at constant angular velocity ω. Such flow is generally described by the Navier-Stokes equations in which the velocity has in a cylindrical coordinate system the components u, v, and w, in the radial, azimuthal, and axial directions, respectively, and the basic simplification reduces to the symmetry of solution about the z-axis.

Denoting dimensional coordinates by r° and z°, density by ρ, pressure by p and kinematic viscosity by ν, we introduce dimensionless variables and the small parameter, respectively, by

$$r = r^\circ / R, \quad Z = z^\circ / (\varepsilon R), \quad F(r, Z) = u / (r\omega R) \tag{1.1}$$

$$G(r, Z) = v / (r\omega R), \quad H(r, Z) = w / (\varepsilon\omega R)$$

$$P(r, Z) = (p - p_\infty) / (\rho\omega^2 R^2)$$

$$\varepsilon = \mathrm{Re}^{-1/2} = \sqrt{\nu / (\omega R^2)} \tag{1.2}$$

The Navier-Stokes equations and the boundary conditions in variables (1.1) are of the form

$$rF\frac{\partial F}{\partial r} + F^2 - G^2 + H\frac{\partial F}{\partial Z} = -\frac{1}{r}\frac{\partial P}{\partial r} + \frac{\partial^2 F}{\partial Z^2} + \varepsilon^2\left(\frac{\partial^2 F}{\partial r^2} + \frac{3}{r}\frac{\partial F}{\partial r}\right) \tag{1.3}$$

$$rF\frac{\partial G}{\partial r} + 2FG + H\frac{\partial G}{\partial Z} = \frac{\partial^2 G}{\partial Z^2} + \varepsilon^2\left(\frac{\partial^2 G}{\partial r^2} + \frac{3}{r}\frac{\partial G}{\partial r}\right)$$

$$\frac{\partial P}{\partial Z} = \varepsilon^2 \left(\frac{\partial^2 H}{\partial Z^2} - rF\frac{\partial H}{\partial r} - H\frac{\partial H}{\partial Z}\right) + \varepsilon^4 \left(\frac{\partial^2 H}{\partial r^2} + \frac{1}{r}\frac{\partial H}{\partial r}\right)$$
$$r\frac{\partial F}{\partial r} + 2F + \frac{\partial H}{\partial Z} = 0$$
$$F = H = G - 1 = 0 \quad \text{for } Z = 0,\ r < 1 \tag{1.4}$$
$$H = \partial F / \partial Z = \partial G / \partial Z = 0 \quad \text{for } Z = 0,\ r > 1$$
$$F \to 0, \quad G \to 0 \quad \text{for } Z \to \infty$$

It is further assumed that the Kármán solution

$$F = F_0 (Z), \qquad G = G_0 (Z), \quad H = H_0 (Z)$$

is valid for all points with coordinates $r < 1$ and fairly remote from the edge.

At $r > 1$ at some distance from the edge there exists a trail region whose properties are very similar to those of the trail downstream of the trailing edge of a flat plate [5, 6]. In conformity with the cited works we introduce in the analysis the stream function $\Psi (r, Z)$, which defines the flow in radical planes drawn through the disk axis. By the last of Eqs. (1.3) that function is defined by the relationships

$$r^2 F = \partial \Psi / \partial Z, \qquad rH = -\partial \Psi / \partial r \tag{1.5}$$

For analyzing the trail region we shall use the new variable

$$s = r - 1 \tag{1.6}$$

and investigate the range of small s (the specific order of smallness of its values will be defined later). We assume, as in [5], that in the region of the near trail an inner and an outer subregion are distinguished, and that in the process of solution the passing to limit $\varepsilon \to 0$ is accomplished without additional distortion of the coordinate grid, i.e. in "boundary layer" coordinates s and Z.

In the inner part of the trail functions Ψ and G are sought in the form of expansions of the form

$$\Psi = s^{2/3} f_0 (\eta) + s f_1 (\eta) + \ldots \tag{1.7}$$
$$G = 1 + s^{1/3} g_0 (\eta) + s^{2/3} g_1 (\eta) + \ldots, \quad \eta = Z / s^{1/3}$$

Using (1.5) it is possible to obtain from this also the expansions for F and H

$$F = s^{1/3} f_0' (\eta) + s^{2/3} f_1' (\eta) + \ldots \tag{1.8}$$
$$H = {}^1/_3 s^{-1/3} [\eta f_0' (\eta) - 2 f_0 (\eta)] + {}^1/_3 [\eta f_1' (\eta) - 3 f_1 (\eta)]$$

After the introduction in (1.3) of the new argument defined by (1.6) and the substitution into it of expressions (1.7) and (1.8), we collect terms containing multipliers of like powers of s and obtain equations for f_0, g_0, f_1, g_1 etc. Boundary conditions for $\eta = 0$ correspond to conditions (1.4) when $Z = 0$ and $r > 1$, while for $\eta \to \infty$ the conditions of asymptotic joining with the solution of the external region, whose principal term corresponds to the Kármán solution, must be satisfied. By taking into consideration the form of the latter in the neighborhood of point $Z = 0$ it is possible to obtain for f_0 and g_0 the following equations and boundary conditions:

$$f_0''' + {}^2/_3 f_0 f_0'' - {}^1/_3 f_0'^2 = 0, \quad g_0'' + {}^2/_3 f_0 g_0' - {}^1/_3 f_0' g_0 = 0 \tag{1.9}$$
$$f_0(0) = f_0''(0) = g_0'(0) = 0$$
$$f_0 = {}^1/_2 a (\eta + \delta_0)^2, \quad g_0 = b(\eta + \delta_0) \quad \text{for } \eta \to \infty$$
$$a = F_0'(0) = 0.510, \quad b = G_0'(0) = -0.616$$

where the constant δ_0 is determined during the process of numerical solution of the boundary value problem. In this case we obtain $\delta_0 = 1.1165$, $f_0'(0) = 1.0283$ and $g_0(0) = -1.2421$. We also adduce the equations and boundary conditions for f_1 and g_1

$$f_1''' + {}^2/_3 f_0 f_1'' - f_0' f_1' + f_0'' f_1 = -1 \tag{1.10}$$
$$g_1'' + {}^2/_3 f_0 g_1' - {}^2/_3 f_0' g_1 = {}^1/_3 f_1' g_0 - f_1 g_0'$$
$$f_1(0) = f_1''(0) = g_1'(0) = 0$$
$$f_1 = -\frac{1}{6}(\eta + \delta_0)[(\eta + \delta_0)^2 + \delta_1], \quad g_1 = -\frac{b}{6a}\delta_1 \quad \text{for } \eta \to \infty$$

in which the new constant δ_1 is determined in the same manner as δ_0.

In the trail outer region the expansions

$$\Psi = r^2 \Psi_0(Z) + s^{1/3}\Psi_1(Z) + s^{2/3}\Psi_2(Z) + \ldots \tag{1.11}$$
$$G = G_0(Z) + s^{1/3}G_1(Z) + s^{2/3}G_2(Z) + \ldots$$

are valid. The form of functions $\Psi_1(Z)$, $G_1(Z)$, etc. is readily determined by the conditions of joining internal and external expansions. We obtain

$$\Psi_1(Z) = \delta_0 \Psi_0'(Z), \quad \Psi_2(Z) = \frac{\delta_0^2}{2}\Psi_0''(Z) - \frac{\delta_1}{6a}\Psi_0'(Z) \tag{1.12}$$
$$G_1(Z) = \delta_0 G_0'(Z), \quad G_2(Z) = -\frac{\delta_1}{6a} G_0'(Z)$$

Equations (1.3) show that the pressure in the external (and, consequently, also in the internal) region of the trail is $P = O(\varepsilon^2)$. However, for elucidation of the flow structure it is important to establish the form of the principal term of the expression for P. Substituting (1.11) and (1.12) into (1.3), we obtain

$$P = -\frac{2}{9}\delta_0 \varepsilon^2 s^{-5/3}\Phi(Z) + O(\varepsilon^2 s^{-4/3}) \tag{1.13}$$

$$\Phi(Z) = \int_{\infty}^{Z} \Psi_0'^2 \, dZ$$

where the choice of the limits of integration is linked with that the axial velocity perturbation at infinity is equal zero, and that by virtue of the Bernoulli equation, the corresponding pressure perturbation is zero. Determination of successive approximations for the above expression does not present serious difficulties.

2. As noted earlier, the solution in the region of the trail close to the disk edge is valid for small s. However at the edge itself where $s = 0$ that solution has a singularity which is apparent in formulas (1.8) and (1.13) for H and P. The only way of eliminating the singularity is to introduce in the edge neighborhood a subsidiary transition region whose dimensions are of a higher order of smallness with respect to ε. The solution in that region is subject to asymptotic joining in radial direction with the Kármán solution on the one hand, and on the other, with the above solution for the trail region.

In the investigation of flow in the transition region we, first of all, introduce extended coordinates r^* and z^* by formulas

$$s = \varepsilon^{\alpha} r^*, \quad Z = \varepsilon^{\beta} z^* \quad (\alpha, \beta > 0) \tag{2.1}$$

where parameters α and β are to be determined. Inside the transition zone we distinguish the inner and outer regions, calling the sublayer in which viscosity and inertia term of equations are of the same order, the inner region.

The definition of the outer region of the transition zone comprises the passing to limit $\varepsilon \to 0$ in the Navier-Stokes equations for fixed r^* and Z. It is assumed that the asymptotic expansions

$$\begin{aligned} \Psi(s, Z; \varepsilon) &= r^2 \Psi_0(Z) + \varepsilon^k \psi_1(r^*, Z) + \ldots \\ G(s, Z; \varepsilon) &= G_0(Z) + \varepsilon^k h_1(r^*, Z) + \ldots \\ P(s, Z; \varepsilon) &= \varepsilon^m p_1(r^*, Z) + \ldots \end{aligned} \tag{2.2}$$

where the zero subscript again relates to the Kármán solution, are valid in that region.

We make the following assumptions about the exponent of the small parameter ε:

$$\alpha < 1, \quad m = k + 2(1 - \alpha) \tag{2.3}$$

The substitution on these assumptions of (2.2) into the Navier-Stokes equations with subsequent passing to limit $\varepsilon \to 0$ yields

$$\begin{aligned} &\Psi_0'' \frac{\partial^2 \psi_1}{\partial r^* \partial Z} - \Psi_0' \frac{\partial \psi_1}{\partial r^*} = 0 \\ &\Psi_0' \frac{\partial h_1}{\partial r^*} - G_0' \frac{\partial \psi_1}{\partial r^*} = 0, \quad \frac{\partial p_1}{\partial Z} = \Psi_0' \frac{\partial^2 \psi_1}{\partial r^{*2}} \end{aligned} \tag{2.4}$$

The first of Eqs.(2.4) can be integrated, yielding

$$\psi_1 = \Psi_0'(Z)\, D(r^*) \tag{2.5}$$

where the arbitrary additive function of Z is assumed to be identically zero owing to the conditions pf joining with the Kármán solution. Using (2.5) we similarly determine

$$h_1 = G_0'(Z)\, D(r^*) \tag{2.6}$$

$$p_1 = D''(r^*)\, \Phi(Z) \tag{2.7}$$

where $\Phi(Z)$ is the same function as that appearing in (1.13).

The principal term of expansions for the inner viscous sublayer of the transition region can be determined by analogy with (2.2), assuming that there

$$\begin{aligned} \Psi(s, Z; \varepsilon) &= \varepsilon^n \psi_1^*(r^*, z^*) + \ldots \\ G(s, Z; \varepsilon) &= 1 + \varepsilon^q h_1^*(r^*, z^*) + \ldots \\ P(s, Z; \varepsilon) &= \varepsilon^m p_1^*(r^*, z^*) + \ldots \end{aligned} \tag{2.8}$$

The principal terms in (2.8) must join with the principal terms in (2.2) when in the former $z^* \to \infty$ and in the latter $Z \to 0$. This implies that functions ψ_1^* and h_1^* must behave as $az^{*2}/2$ and bz^* respectively. But, then, the realization of joining requires that $n = 2\beta$ and $q = \beta$.

In conformity with the definition of the viscous sublayer inertia and viscous forces must be of the same order, which implies that condition $\alpha = 3\beta$ must be satisfied. The solution for the outer region of the transition zone when $r^* \to \infty$ must join with the solution for the outer region of the trail. For this it is necessary that

$$D(r^*) \to \delta_0 r^{*1/3} \quad \text{при } r^* \to \infty \tag{2.9}$$

Taking into account (2.9) we find that the joining of expressions for Ψ (2.2) and (1.11) yields $k = \alpha / 3 = \beta$.

The additional assumption that $m < 2(1 - \beta)$ shows that in the asymptotic approximation pressure p_1^* may be taken as constant across the viscous sublayer. Taking into account the requirement for joining with solution (2.7) for the pressure in the outer region sublayer we obtain the general expression

$$p_1^* = \Phi(0) D''(r^*) \tag{2.10}$$

The last of the formulas required for the determination of exponents in expansions (2.2) and (2.8) follows from the condition of equality of order for terms containing pressure and viscosity and inertia terms in the viscous sublayer. That condition yields $m - \alpha = -\beta$, and, with allowance for (2.3) makes it possible to determine β. to which are linked all other exponents. The obtained values $\alpha = {}^6/_7$, $\beta = {}^2/_7$, $k = {}^2/_7$, $m = {}^4/_7$, $n = {}^4/_7$, and $q = {}^2/_7$ fully correspond to the previously made assumptions, and α and β determine the dimensions of the flow transition zone at the disk edge.

With known exponent α it is, also, possible to determine with greater precision the range of values for s, for which the solution derived in Sect. 1 for the trail region is valid. If, for example, it is required that the desired accuracy (1.7) and (1.11) is ensured by the principal terms, it is necessary to set $s^{4/3} = O(\varepsilon^{6/7})$, or $s = O(\varepsilon^{9/14})$ When a greater number of terms is taken into account in the indicated expansions, the result will be different. We denote by i the number of the approximation whose contribution is negligibly small, and for this more general case set

$$s = O(\varepsilon^{\gamma}), \quad \gamma = {}^{18}/_{7}\,(3 + i)^{-1} \tag{2.11}$$

3. The problem of flow pattern in the neighborhood of the disk edge reduces in essence to the solution of equations for the inner sublayer of the transition zone. In conformity with the results obtained in Sect. 2 these equations and boundary conditions are of the form

$$\frac{\partial\psi_1^*}{\partial z^*}\frac{\partial^2\psi_1^*}{\partial r^*\partial z^*} - \frac{\partial\psi_1^*}{\partial r^*}\frac{\partial^2\psi_1^*}{\partial z^{*2}} = -\Phi(0)\,D'''(r^*) + \frac{\partial^3\psi_1^*}{\partial z^{*3}} \tag{3.1}$$

$$\frac{\partial\psi_1^*}{\partial z^*}\frac{\partial h_1^*}{\partial r^*} - \frac{\partial\psi_1^*}{\partial r^*}\frac{\partial h_1^*}{\partial z^*} = \frac{\partial^2 h_1^*}{\partial z^{*2}}, \quad \frac{\partial p_1^*}{\partial z^*} = 0$$

$$\psi_1^* = \partial\psi_1^* / \partial z^* = h_1^* = 0 \quad \text{for } z^* = 0,\ r^* < 0 \tag{3.2}$$

$$\psi_1^* = \partial^2\psi_1^* / \partial z^{*2} = \partial h_1^* / \partial z^* = 0 \quad \text{for } z^* = 0,\ r^* > 0$$

$$\psi_1^* \to az^{*2} / 2, \quad h_1^* \to bz^* \quad \text{for } r^* \to -\infty$$

$$\psi_1^* \to (a / 2)\,[z^* + D(r^*)]^2, \quad h_1^* \to b\,[z^* + D(r^*)] \quad \text{for } z^* \to \infty$$

$$\psi_1^* \to r^{*2/3} f_0(\eta), \quad h_1^* \to r^{*1/3} g_0(\eta) \quad \text{for } r^* \to \infty \quad (\eta = z^* / r^{*1/3})$$

where functions f_0 and g_0 satisfy Eqs. (1.4).

It is not possible to affirm unconditionally that a solution of problem (3.1), (3.2) exists and has a physical meaning. It is, however, possible to refer to the numerical solution of the similar problem of the flat plate [7] and, also, attempt by following Stewartson [3] to determine the properties of the considered problem for fairly high values of $|r^*|$. If such analysis does not reveal contradictions with input assumptions and would, also, indicate the possibility of determination of integral characteristics of the flow, we shall consider that it confirms the assumption of existence of solution of problem (3.1), (3.2).

Let us, first, consider the region of high positive values or r^*, i.e. $r^* \to \infty$. The limit form of solution in that region is determined by conditions (3.2), owing to that form it is advisable to introduce expansions of the form

$$\psi_1^* = r^{*2/3} f_0(\eta) + r^{*(2-n)/3} f_n(\eta) + \ldots, \tag{3.3}$$

$$h_1^* = r^{*1/3} g_0(\eta) + r^{*(1-n)/3} g_n(\eta) + \ldots$$

where the dots denote terms with higher powers of r^{*-1}. Substituting these into (3.1) we obtain equations for f_n and g_n and from (3.2) we have their boundary conditions

$$f_n''' + \frac{2}{3} f_0 f_n'' - \frac{2-n}{3} f_0' f_n' + \frac{2-n}{3} f_0'' f_n = A_n \tag{3.4}$$

$$g_n'' + \frac{2}{3} f_0 g_n' - \frac{1-n}{3} f_0' g_n = \frac{1}{3} [f_n' g_0 - (2-n) f_n g_0'] + B_n$$

$$f_n(0) = f_n''(0) = g_n'(0) = f_n''(\infty) = g_n'(\infty) = 0$$

The lowest value of n at which $A_n \neq 0$ is seven and $A_7 = {}^{10}/_{27}\delta_0\Phi(0)$ and $B_7 = 0$. The first of Eqs. (3.4) can, however, have eigenfunctions with required properties at lower values of n. Thus, for $n = 3$ when $A_3 = B_3 = 0$, system (3.4) has the solution

$$f_3 = C(2f_0 - \eta f_0'), \quad g_3 = C(g_0 - \eta g_0') \tag{3.5}$$

where C is a constant that cannot be determined by asymptotic analysis.

Using solution (3.5) it is possible to establish a more precise form of function $D(r^*)$ at considerable positive r^* and determine expressions for the components of velocity in the disk plane and, also, to obtain a more precise expression for pressure p_1^*. We have

$$D(r^*) = \delta_0 r^{*1/3} + C\delta_0 r^{*-2/3} + \ldots \tag{3.6}$$

$$(\partial\psi_1^* / \partial z^*)_{z^*=0} = r^{*1/3} f_0'(0) + Cr^{*-8/3} f_0'(0) + \ldots \tag{3.7}$$

$$h^*_{1z^*=0} = r^{*1/3} g_0(0) + Cr^{*-2/3} g_0(0) + \ldots$$

$$p_1^* / [\Phi(0)\delta_0] = -{}^2/_9 r^{*-5/3} + {}^{10}/_9 C r^{*-8/3} + \ldots$$

We pass to the invesitgation of region $r^* \to -\infty$, and introduce expansions entirely similar to (3.3), except that f_i and g_i, are substituted in it for φ_i and ζ_i Taking into account that $\varphi_0 = a\eta^2/2$ and $\zeta_0 = b\eta$, for φ_n and ζ_n we obtain equations with the boundary conditions

$$\varphi_n''' + \frac{a}{3}\eta^2\varphi_n'' + \frac{a(n-2)}{3}\eta\varphi_n' - \frac{a(n-2)}{3}\varphi_n = M_n \tag{3.8}$$

$$\zeta_n'' + \frac{a}{3}\eta^2\zeta_n' + \frac{a(n-1)}{3}\eta\zeta_n = \frac{b}{3}[\eta\varphi_n' + (n-2)\varphi_n] + N_n$$

$$\varphi_n(0) = \varphi_n'(0) = \zeta_n(0) = \varphi_n''(\infty) = \zeta_n'(\infty) = 0$$

that follow from (3.2). For any n the solution of Eqs. (3.8) can be expressed in terms of some combinations of degenerate hypergeometric functions. However, unlike in the case of Eqs. (3.4), it is not possible to derive a solution which for $M_n = 0$ satisfies all boundary conditions. The value of n for which $M_n \neq 0$, is determined by the form of the principal term in the expression of p_1^* when $r^* \to -\infty$.

Let that term be of the form

$$p_1{}^* \sim |\, r^* \,|^{-q} = |\, s \,|^{-q} \varepsilon^{\alpha q} \tag{3.9}$$

For determining exponent q we reason as follows. Owing to the symmetry of dimensions of the transition zone the value of $|s|$ in formula (3.9) must correspond to the estimate defined by (2.11) for the boundary of that zone. Setting $i = 1$, we obtain $|s| \sim \varepsilon^{\gamma}$ and $\gamma = {}^9/_{14}$. The quantity ε^m, which links p and $p_1{}^*$ in conformity with (2.8) is, by definition, small, and we shall assume that its square can be neglected. Then, if formula (3.9) is to yield a negligibly small quantity p the following relation must be satisfied:

$$2m = m + \alpha q - \gamma q$$

Substitution into this of known values of m and α and also the indicated above value of γ, yields $q = {}^8/_3$, or

$$D(r^*) = C_1 r^{*-{}^2/_3} + \ldots, \quad p_1{}^* = {}^{10}/_9 C_1 \Phi(0) r^{*-{}^8/_3} + \ldots \tag{3.10}$$

where we have again the constant C_1, which can be determined only in the course of solving the problem as a whole. Using (3.10) we obtain $n = 10$, $M_{10} = {}^{80}/_{27} C_1 \Phi(0)$ and $N_{10} = 0$, and Eqs. (3.8) are now satisfied by the following expressions

$$\varphi_{10} = -\frac{10}{9a} C_1 \Phi(0) \Big\{ 1 - 7 \left(\frac{a}{9}\right)^{{}^2/_3} \frac{\Gamma({}^4/_3)}{\Gamma({}^5/_3)} \eta^2 \times \Big[U\left(\frac{1}{3}, \frac{5}{3}, -\frac{a}{9}\eta^3\right) + \frac{a}{18}\eta^3 U\left(\frac{4}{3}, \frac{8}{3}, -\frac{a}{9}\eta^3\right) + \frac{a^2}{567}\eta^6 U\left(\frac{7}{3}, \frac{11}{3}, -\frac{a}{9}\eta^3\right)\Big]\Big\} \tag{3.11}$$

$$\zeta_{10} = \frac{70b}{9a^2}\left(\frac{a}{9}\right)^{{}^2/_3} \frac{\Gamma({}^4/_3)}{\Gamma({}^5/_3)} C_1 \Phi(0)\, \eta \times \Big[2U\left(\frac{1}{3}, \frac{5}{3}, -\frac{a}{9}\eta^3\right) + \frac{7a}{18}\eta^3 U\left(\frac{4}{3}, \frac{8}{3}, -\frac{a}{9}\eta^3\right) + \frac{22a^2}{567}\eta^6 U\left(\frac{7}{3}, \frac{11}{3}, -\frac{a}{9}\eta^3\right) + \frac{a^3}{729}\eta^9 U\left(\frac{10}{3}, \frac{14}{3}, -\frac{a}{9}\eta^3\right)\Big]$$

where functions $U(\alpha, \beta, x)$ correspond to the so-called second form of solutions of the degenerate hypergeometric equation [8].

Away from the disk plane, i.e. when $\eta \to -\infty$, functions $d\varphi_{10} / d\eta$ and ζ_{10} tend to a constant limit whose presence makes it possible to determine the following term of the asymptotic expansion of $D(r^*)$:

$$D(r^*) = C_1 r^{*-{}^2/_3} - 5C_1 \frac{1}{a^2}\left(\frac{a}{9}\right)^{{}^2/_3} \frac{\Gamma({}^4/_3)}{\Gamma({}^5/_3)} \Phi(0)\, r^{*-3} + \ldots \tag{3.12}$$

From this with the use of (2.10) it is possible to obtain the second term of the expansion for $p_1{}^*$ and, then repeat the whole described procedure.

The friction stress at the disk surface and the general expression for the coefficient of the moment of friction forces are of the form

$$\tau_{z\theta} = \rho\nu\left(\frac{\partial v}{\partial z^\circ}\right)_w = r\rho\omega^2R^2\varepsilon\left(\frac{\partial G}{\partial Z}\right)_w = r\rho\omega^2R^2\varepsilon\left(\frac{\partial h_1{}^*}{\partial z^*}\right)_w = \tag{3.13}$$

$$r\rho\omega^2R^2\varepsilon\left[b + \left(\left(\frac{\partial h_1{}^*}{\partial z^*}\right)_w - b\right)\right]$$

$$C_M = -\frac{2}{\rho\pi\omega^2R^5}\int_0^R 2\pi r^{\circ 2}\tau_{z\theta}\,dr^\circ = -b\varepsilon - 4\varepsilon^{1+\alpha}\int^0 \left[\left(\frac{\partial h_1{}^*}{\partial z^*}\right)_w - b\right]dr^* \tag{3.14}$$

Use is made here of the fact that the region of existence of the last term in brackets of the right-hand side of (3.13) corresponds to the region of determination of r^*. In conformity with expansion of the type (3.3) the principal term of the integrand in (3.14) for $r^* \to -\infty$ is

$$r^{*-10/3}\zeta_{10}'(0) = \frac{140b}{9a^2}\left(\frac{a}{9}\right)^{2/3}\frac{\Gamma^2(4/3)}{\Gamma^2(5/3)}C_1\Phi(0)\,r^{*-10/3}$$

The last expression represents a constant multiplied by a high negative power of r^*, which makes it possible to consider the integral in the right-hand side of (3.14) as convergent. The second term in the right-hand side of (3.14) represents the addition to coefficient C_M, due to the finiteness of disk. It is proportional to $\varepsilon^{13/7}$ and, evidently, has the highest value among other possible corrections to the coefficient of the moment of friction forces of an infinite disk. The complete solution of the stated problem requires numerical integration of system (3.1) with boundary conditions (3.2).

The aim of this work was the qualitative clarification of the stream pattern in the neighborhood of the disk edge, and was achieved without resorting to numerical calculations. However, taking all aspects into account, it must be said that the solution which completely satisfies Eqs.(3.1), as well as conditions (3.2) may, nevertheless, contain discontinuities of the unknown functions or their derivatives when $r^* = 0$. If the presence of such singularities is confirmed in the course of calculations, it will be necessary to introduce for their elimination in the close neighborhood of the edge one more asymptotic region, which should not affect the validity of the obtained here results. To estimate the central region dimensions the reasoning in [4] applied to the case of the flat plate may be repeated here, to find that the order of dimension $O(\varepsilon^{3/2})$ in the radial and axial directions is of the same order, and that the flow in that region is defined by the complete Navier-Stokes equations.

We note in conclusion that the asymptotic multilayer pattern of flow, similar to that described here, appears in investigations of a fairly wide class of problems with stepwise variation of boundary conditions. In addition to previously cited publications [3, 4] we may point out other investigations of flow in the neighborhood of edges [9, 10], and also [11-13] devoted to flows close to boundary layer separation points in an incompressible stream and in a supersonic gas stream. All these investigations have much in common between themselves and with the present investigation as regards the principles of asymptotic analysis, however, the results presented here reveal certain singularities not found in other edge flows, such as: the substantially different order of basic dimensions than in the case of a plate, the zone of inviscid flow at variable pressure, etc.

REFERENCES

1. Kármán, T. von, Über laminare und turbulente Reibung. ZAMM, Vol. 1, № 4, 1921.
2. Cochran, W.G., The flow due to a rotating disk. Proc. Cambridge Philos. Soc., Vol. 30, № 3, 1934.
3. Stewartson, K., On the flow near the trailing edge of a flat plate II. Mathematika, Vol. 16, Pt 1, 1969.
4. Messiter, A.F., Boundary layer flow near the trailing edge of a flat plate. SIAM J. Appl. Math., Vol. 18, № 1, 1970.
5. Goldstein, S., Concerning some solutions of the boundary layer equations in hydrodynamics. Proc. Cambridge Philos. Soc., Vol. 26, № 1, 1930.
6. Leslie, L.M., The wake of a finite rotating disk. J. Austral. Math. Soc. Vol. 13, Pt 3, 1972.
7. Jobe, C.E. and Burggraf, O.R., The numerical solution of the asymptotic equations of trailing edge flow. Proc. Roy. Soc., Ser. A, Vol. 340, № 1620, 1947.
8. Slater, L.J., Degenerate Hypergeometric Functions, Moscow, V. Ts. Akad. Nauk SSSR, 1966.
9. Brown, S.N. and Stewartson, K., Trailing edge stall. J. Fluid Mech., Vol. 42, Pt 3, 1970.
10. Guiraud, J.-P., Ecoulement décollé au voisinage du bord de fuite d'une aile mincetridimensionelle. J. Mech., Vol. 13, № 3, 1974.
11. Sychev, V.V., On laminar separation. Izv. Akad. Nauk SSSR, M. Zh. G., № 3, 1972.
12. Neiland, V. Ia., On the theory of laminar boundary layer separation in a supersonic stream. Izv. Akad. Nauk SSSR, M. Zh. G., № 4, 1969.
13. Neiland, V. Ia., The flow behind the separation point of the boundary layer in a supersonic stream. Izv. Akad. Nauk SSSR, M. Zh. G., № 3, 1971.

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